Exercises on dynamic programming and optimal control
Doctoral Advanced Macroeconomics, Fall 2024
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Due on Jan/05/2025

1. We are trying to solve the following growth model:

$$\text{Lifetime utility}: U = \sum_{t=0}^{\infty} (\beta)^t \log c_t, \tag{1}$$

$$\text{subject to} \quad k_{t+1} = A k_t^{\alpha} - c_t. \tag{2}$$

(a) i. You are asked to do the "guess-and-verify" exercise. First, let us guess the **value function** and **policy function** of the Bellman Equation for the dynamic programming problem above as:

$$\text{value function}: V(k_t) = \lambda + \xi \log k_t, \tag{3}$$

$$\text{policy function}: k_{t+1} = \pi(k_t) = \gamma A k_t^{\alpha}, \tag{4}$$

Then, you should use the Euler equations in your lecture notes to prove the following statements

$$\xi = \frac{\alpha}{1-\alpha\beta}, \quad \lambda = \frac{\log[A(1-\alpha\beta)]}{1-\beta} + \frac{\alpha\beta\log(A\alpha\beta)}{(1-\alpha\beta)(1-\beta)}, \quad \gamma = \alpha\beta.$$

ii. Given $\beta = 0.99$, $\alpha = 0.2$, $A = 2$. Based on your results in (a), draw the value function and policy function out, using *your favorite software*. What is the steady-state capital stock k^* and consumption c^*?

(b) Restate the problem above in the form of Problem A2 and Problem A3 as in the lecture notes.

(c) Do the following steps (notice: make sure that you know why we are doing the following steps! If not, you should double-check the lecture notes and recall that the Bellman function actually constructs a contraction mapping!):

- Define the maximum and minimum values k can take as a 90% deviation from the steady state value of k (we are not interested in all feasible value of k). Next, create a vector of length $N = 1000$ as the grid values k can take, bounded by the minimum and maximum values you have just calculated. Let us denote that vector k with elements $k(1) < k(2) < \cdots < k(N)$, with $k(1)$ equal to the minimum value and $k(N)$ equal to the maximum value.

- Pick a small value ϵ as the convergence criterion (any number you think sensible). A number too small will take you forever to run the program and a number too big will give you inaccurate estimates. Let the initial guess for the value function to be $V_0(k) \equiv 0$ for any k.
- For each $i = 1, \cdots, N$, find the $k(j)$ that maximizes $\log\left[2k(i)^{0.2} - k(j)\right]$ (recall that $V_0 \equiv 0$). Make sure that you do not pick a $k(j)$ that makes consumption negative, for all $i = 1, \cdots, N$. Then keep the maximum value as $V_1(i)$ and memorize the "position" j (i.e., which grid value of capital you have picked above while solving the maximization problem).

  After you have done the maximization problem above for all $i = 1, \cdots, N$, you should have a $N \times 1$ vector of maximum values V_1 and a $N \times 1$ vector of policy π_1, which contains the "position" of the grid value of capital you have picked. The policy π_1 tells you what next period capital you should pick given the current capital: $k_{t+1} = \pi_1(k_t)$.
- For each $i = 1, \cdots, N$, find the $k(j)$ that maximizes $\log[2k(i)^{0.2} - k(j)] + \beta V_1(j)$. Then keep the maximum value as $V_2(i)$ and memorize the "position" j (i.e., which grid value of capital you have picked above while solving the maximization problem).

  After you have done the maximization problem above for all $i = 1, \cdots, N$, you should have a $N \times 1$ vector of maximum values V_2 and a $N \times 1$ vector of policy π_2, which contains the "position" of the grid value of capital you have picked. The policy π_2 tells you what next period capital you should pick given the current capital: $k_{t+1} = \pi_2(k_t)$.
- Repeat the above step many times, until

$$\max_j\{|V_n(j) - V_{n-1}(j)|\} < \epsilon. \tag{5}$$

  That is, the iterative algorithm continues until the largest absolute difference between the corresponding elements for the two value functions is less than ϵ.

  Your program has converged to the fixed point!

  Now, treat V_n as your value function and π_n as your policy function obtained from the iterative method above. Plot the two functions with the grid values of k on the x-axis. Are the functions the same as those you found in (a)?

(d) Repeat (b) with a lower discount rate $\beta = 0.8$. How does the slope of the policy function change? What does that mean *in words*?

(e) Repeat (b) with the CRRA utility function $\mu(c_t) = \frac{c_t^{0.5} - 1}{0.5}$ and $\beta = 0.99$. What difference does the new functional form make?

(f) Repeat (b) with a bigger $\epsilon = 0.001$. Can you find any differences?

2. Consider the problem below:

$$\max_{\{k(t),c(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c(t)$$
$$\text{subject to} \quad k(t+1) = k(t)^{\alpha} - c(t),$$
$$k(0) > 0, \beta \in (0,1)$$

   (a) Method 1: Guess the policy function as $\pi(x) = \gamma x^{\alpha}$, and verify your guess by determining the value of γ. (economic intuition?)

   (b) Method 2: Guess the value function as $V(x) = \lambda + \xi \log x$, and verify your guess by determining the values of λ and ξ.

   - You should find that the two methods above are equivalent.

3. Consider the following problem:

$$\max_{[c(t),a(t)]_{t=0}^{1}} \int_0^1 e^{-\rho t} u(c(t)) \mathrm{d}t, \tag{6}$$
$$\text{subject to} \quad \dot{a}(t) = ra(t) + \omega - c(t), \quad a(0) = a_0, \ a(1) = 0. \tag{7}$$

   where r and ω are exogenously defined constants.

   (a) Deduce the Euler-Lagrange equation for the problem above.

   (b) Rearrange your result above to give the Euler equation usually used in your textbooks, $\frac{u''(c(t))\dot{c}(t)}{u'(c(t))} = \rho - r$, namely, along the household's optimal path, the growth rate of its marginal utility of consumption should be equal to the gap between the discount rate ρ and interest rate r.

   (c) Use the Pontryagin's Maximum Principle (Theorem 4 in your lecture notes) to get the same results.

   (d) Given $u(c) = \log(c)$, can you ***solve*** the problem above? What if $u(c) = \left[\theta - e^{-\beta c(t)}\right]$?