Advanced Macroeconomics
Midterm Exam I (Open-Book)
Undergraduate Economics Program, HUST
Thursday, April/09/2015

Name: ________________________ **Student ID:**________________________

1. (5+15+15=35 points) After introducing a government into the basic Solow model, equation (1.15) in the textbook becomes:

$$\begin{aligned}\dot{K}(t) &= I(t) - \delta K(t) \\ &= [Y(t) - C(t) - G(t)] - \delta K(t),\end{aligned}$$

where $Y(t)$ denotes the output at time t, $C(t)$ is consumption, and $G(t)$ represents the government spending. Further assume that $G(t)$ is determined by $G(t) = \sigma Y(t)$.

(a) Is it reasonable to assume that $C(t) = (1 - s)Y(t)$? If no, explain why. If yes, what does this condition mean?

(b) Suppose that government spending is partly supported by private consumption, so that $C(t) = (1 - s - \lambda\sigma)Y(t)$, where $\lambda \in [0, 1]$ is a constant factor. If no government spending adds to the capital stock, what is the effect of higher government spending (in the form of higher σ) on the steady-state k and c of the Solow model? (Hint: Portion λ of government spending comes out of consumption, so the rest must come out of investment.)

(c) Now suppose that a fraction ϕ of $G(t)$ is invested in the capital stock, so that total investment at time t is given by

$$I(t) = [s - (1 - \lambda)\sigma + \phi\sigma]\, Y(t).$$

Show that if ϕ is sufficiently high, the steady-state capital per effective labor (k^*) will increase in σ. What is the economic intuition behind this result?

2. (17+18=35 points) The following questions are based on the specific Ramsey-Cass-Koopmans model in the textbook.

(a) Consider Figure 2.4 in the textbook. Suppose that at time $t = 0$, given $k(0)$, instead of jumping immediately to point F on the saddle path, a household chooses $c(0)$ such that its initial consumption-capital bundle is depicted by point D in the figure, and then the household keeps $k(t) = k(0)$ for all $t \leq \Delta t$ and increases its consumption gradually and smoothly for all $t \leq \Delta t$ and make sure that bundle $(c(\Delta t), k(\Delta t))$ is exactly point F. From then on, the household just follows the saddle path illustrated in the figure to approach point E. Does this arrangement violate the household's budget constraint? If yes, explain why. If no, explain why and further check if the arrangement solves the household's maximization problem.

(b) Now consider a world consisting of a list of closed economies, each of them is well characterized by a Ramsey-Cass-Koopmans model in the textbook. Further assume that there is no technological growth in all countries ($A_i(t) = \bar{A}(0)$ for any time t and country i), all countries are almost identical except that different countries have different ρ's. Production technology is defined by a C-D function $y = f(k) = k^{\alpha}$. Suppose that for countries i and j, we have $\rho_i = \gamma\rho_j$, where $\gamma > 0$ is a constant. If the capital markets in all countries are completely competitive, and the capital share of income in all countries is fixed at $\frac{1}{3}$, how will you predict y_i^*/y_j^*? (Hint: What is the relationship between the capital share of income and the parameter α in the C-D function?)

3. (5+10+7+8=30 points) Suppose an economy is depicted by the Diamond Overlapping-Generations model with $\theta = 1$ and $f(k) = k^{\alpha}$.

(a) What is the CRRA utility function form under condition $\theta = 1$?

(b) Characterize the steady-state, and show that it is globally stable.

(c) What is the effect of an increase in g on the steady-state? Provide an intuition for this result.

(d) What is the effect of an increase in ρ on the steady-state? Provide an intuition for this result.

Solutions and Hints

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I did not have enough time to double-check this file while writing it. So please do me a favor by pointing out the mistakes and typos, if any, that I have made. (just send an email to yiming@hust.edu.cn)

1.(a) $C(t) = (1-s)Y(t)$ simply means that consumption level has nothing to do with government spending, or that government spending does not crowd out consumption at all, or that government spending is fully supported by taxes on investment, rather than taxes on consumption.

Points are earned as long as you manage to make any statement above, and induce your answer to the question based on your explanation.

1.(b) The rest of government spending, portion $(1-\lambda)$ must come out of investment. This leads to

$$\begin{aligned} \dot{K}(t) &= I(t) - \delta K(t) \\ &= [s - \sigma(1-\lambda)]\, Y(t) - \delta K(t). \end{aligned} \tag{1}$$

Combining (1) with equations (1.17) in textbook, we have

$$\dot{k}_(t) = [s - \sigma(1-\lambda)]\, f(k(t)) - (n+g+\delta)k(t), \tag{2}$$

which is simply the revised version of equation (1.18) in your textbook. (2) further implies that the unique steady state of this model is associated with a unique k^* that makes sure the right-hand-side of (2) equals 0.

Next step is to show $\frac{\partial k^*}{\partial \sigma} < 0$. you can achieve this by discussing the monotonicity of the right-hand-side of (2) in k, or by showing a movement in the figure similar to Figure 1.4 in textbook, or by formally adopting the Implicit Function Theorem here. All methods get full points.

Based on the results above, it is straightforward to investigate $c^* = (1-s-\lambda\sigma)y^* = (1-s-\lambda\sigma)f(k^*) \Rightarrow \frac{\partial c^*}{\partial \sigma} < 0$.

If the government spending just crowds out consumption and investment, and never gives back, then an increase in G hurts both investment and consumption in the end.

1.(c) Now the new k^* solves

$$[s - \sigma(1-\lambda-\phi)]\, f(k(t)) - (n+g+\delta)k(t) = 0. \tag{3}$$

Adopting the Implicit Function Theorem to (3) at $k = k^*$ gives us

$$\frac{\partial k^*}{\partial \sigma} = \frac{(1-\lambda-\phi)f(k^*)}{[s-\sigma(1-\lambda-\phi)]\, f'(k^*) - (n+g+\delta)}. \tag{4}$$

The denominator in the right-hand-side is of course negative (guaranteed by conditions $f' > 0, f'' < 0, f'(0) = \infty, f'(\infty) = 0$). But if ϕ is sufficiently high, specifically speaking, when $\phi > 1-\lambda$, the numerator of the right-hand-side is also negative, so $\frac{\partial k^*}{\partial \sigma} > 0$.

Extra question: Does the sign of $\frac{\partial c^*}{\partial \sigma}$ become ambiguous in this case? If the government spending mainly consists of public investments, an increase in G can lead to a higher steady-state capital level. Furthermore, it may occur that an increase in σ increases both k^* and c^* (this is the case if $(1-s-\lambda\sigma)f'(k^*)\frac{\partial k^*}{\partial \sigma} - \lambda f(k^*) > 0$). Recalling the Keynesian Economics you learned in Intermediate Macroeconomics course?

Easy enough huh?

2.(a) No, the arrangement does not violate the household's budget constraint. Intuitively, this is because the path $D \Rightarrow F \Rightarrow E$ is always taking less (or no more) consumption than path $F \Rightarrow E$ at any time. Since we already know that the latter is on the saddle path, and thus satisfies the budget constraint, the former must also satisfy it.

To answer the question in a formal way, you can try the following steps. First, denote by $path1 = \{c(t), k(t)\}_{t=0}^{\infty}$ the path $F \Rightarrow E$. Similarly, path $D \Rightarrow F \Rightarrow E$ is denoted by $path2 = \{\tilde{c}(t), \tilde{k}(t)\}_{t=0}^{\infty}$. Obviously, we have:

$$k(0) = \tilde{k}(\tau),\ \forall \tau \in [0, \Delta t], \tag{5}$$

$$\tilde{c}(\tau)\ and\ c(\tau)\ are\ non-decreasing\ in\ \tau,\ \forall \tau \geq 0, \tag{6}$$

$$\tilde{c}(\tau) < c(0) = \tilde{c}(\Delta t),\ \forall \tau \in [0, \Delta t), \tag{7}$$

$$\tilde{c}(\tau) \leq c(\tau - \Delta t),\ \tilde{k}(\tau) \leq k(\tau - \Delta t),\ \forall \tau > \Delta t. \tag{8}$$

For budget constraint, let us take the version of equation (2.14) in textbook. Suppose a household taking $path1$. further use $LHS(path1)$ and $RHS(path1)$ to denote the values of the left-hand-side and the right-hand-side of (2.14) , respectively. $LHS(path2)$ and $RHS(path2)$ are analogously defined. Obviously, conditions (7) and (8) give us $LHS(path2) < LHS(path1)$. For the right-hand-side values, it is worth noting that the single household's wage does not depend on its path chosen, instead, it is determined by the whole economy, and as an infinitesmal part, its choice has no effect on the whole economy at all. This in turn gives us $RHS(path1) = RHS(path2)$. Combining this equality and the inequality above, we thus have $LHS(path2) < RHS(path2)$. □

Extra question: If either $path1$ or $path2$ must be chosen not by just a single household, but by all the households in the economy, does the result still hold? Yes, actually we have $LHS(path2) = LHS(path3) \leq RHS(path3) < RHS(path2)$. (hint: $w = f(k) - kf'(k)$, so $\frac{\partial w}{\partial k} \gtrless 0$? How to use this result on the right-hand-side of (2.14)? Extend the saddle point in Figure 2.4 to the lower-left corner, which point should be identified as the starting point of $path3$?)

We then move on to answer the second question. The arrangement does not maximize the household's utility. To see this, just note that

$$U(path1) = B\int_{t=0}^{\infty} e^{-\beta t}u(c(t))dt > U(path2) = B\int_{t=0}^{\infty} e^{-\beta t}u(\tilde{c}(t))dt, \quad (9)$$

where the inequality comes from (7) and (8). Given that $path1$ satisfies the budget constraint anyway, $path2$ is inefficient: It does not solve the utility maximization problem, although it does not break the budge constraint requirement.

Further thinking: Does inefficiency of $path2$ occur because it violates condition (2.20) in textbook? Not necessarily! To see this point, just assume that from time $t = 0$ to $t = \Delta t$, $c(t)$ grows an a constant rate $\Lambda = \frac{f'(k(0)) - \rho - \theta g}{\theta}$, make $\Delta t = \frac{\ln c(0) - \ln \tilde{c}(0)}{\Lambda}$ (this is, of course, a special example of the arrangements introduced in the problem above). Does this arrangement satisfy (2.20)?

So, what has been violated? (2.25)! From $t = 0$ to $t = \Delta$, at each time point, a part of output must be wasted — it is not consumed, neither is it invested, the household just throws it away! This is how the non-maximization problem happens.

I admit that 2(a) is somewhat complicated. But again, it is not diffigult to give correct answers purely based on your economic intuition and your understanding of the saddle path.

2.(b) Equation (2.24) in textbook tells us that, in the steady state we must have

$$f'(k_\tau^*) - \rho_\tau - \theta g = 0, \quad \forall \tau \in \{i, j\}. \quad (10)$$

Combining $g = 0$ and $f(k) = k^\alpha$ with the result above leads to

$$\begin{aligned}\gamma &= \frac{\rho_i}{\rho_j} \\ &= \frac{f'(k_i^*)}{f'(k_j^*)} \\ &= \left(\frac{k_i^*}{k_j^*}\right)^{\alpha-1} \qquad (11)\end{aligned}$$

This in turn leads to

$$\ln k_i^* - \ln k_j^* = \frac{\ln \gamma}{\alpha - 1}. \qquad (12)$$

As we illustrated **in details** in class, we know that if the capital share is fixed at α, we should have

$$\frac{\ln y_i^* - \ln y_j^*}{\ln k_i^* - \ln k_j^*} = \alpha. \qquad (13)$$

Substituting $\alpha = \frac{1}{3}$ and (12) into (13) gives us the result:

$$\begin{aligned}\frac{y_i^*}{y_j^*} &= e^{\frac{\alpha}{\alpha-1}\ln\gamma} \\ &= \frac{1}{\sqrt{\gamma}}. \qquad (14)\end{aligned}$$

Intuition: Analogous to that of d.(d).

3.(a) The original version of a CRRA utility function is $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, at $\theta = 1$, the numerator and denominator are both 0. Applying the L'Hôpital's rule here gives us

$$\lim_{\theta\to 1} u(c) = \frac{e^{(1-\theta)\ln c} \cdot (-\ln c)}{-1} = \ln c.$$

Although the textbook takes the simplified version of CRRA function directly, I emphasized its original definition in class for **multiple times**.

3.(b) Section 2.10 in textbook, subsection "Logarithmic and Cobb-Douglas Production".

3.(c) g - growth rate of knowledge. $g \uparrow \Rightarrow$ marginal productivity of capital increases $\Rightarrow$ there is less saving need in the steady state $\Rightarrow k^* \downarrow$. (Think of this scenario: Suppose g is **extremely** large, so the production in next period is pretty high even if other inputs (besides A) reduce, are individuals still willing to save as much as that in a small-g scenario?)

3.(d) ρ - discount rate. $\rho \uparrow \Rightarrow$ value next period's consumption less $\Rightarrow$ save less $\Rightarrow k^* \downarrow$.
Wow...It sounds like 25/30 points for free!